International Journal of Theoretical Physics, Vol. 16, No. 8 (1977), pp. 591-601

An $O(3, 1)$ Approach to Maxwell's Equations

Y. REZVAN

Department of Physics, University of Azarabadgan, Tabriz, Iran

M. SAMIULLAH[1]

International Centre for Theoretical Physics, Trieste, Italy

Received: 25 *January* 1977

Abstract

Using the principal series representations of the Lorentz group, a method parallel to that of Gelfand and Yaglom is suggested to obtain Maxwell's equations, which dispenses with the arbitrary introduction of a degenerate transformation with respect to which the photon equations are invariant. The method also gives subsidiary conditions which, in conjunction with the masslessness of the particle, yield the Lorentz condition and the correct values of photon polarization.

1. *Introduction*

In analyzing and interpreting the experimental data using the homogeneous Lorentz group partial wave analysis due to Delbourgo, Salam, and Strathdee (DSS) (1967), the exchanged trajectory (in the t channel) for which the corresponding physical particle has spin 1, must be assigned two values (Raghavan and Samiullah, 1969; and references therein) of $|\dot{J}_0|$ ($\dot{J}_0$ is M in Toller's language), i.e., 0 and 1. Since the DSS formalism employs only the absolute values of $\dot{J}_0$, in fact, the values associated with such a particle are = 0, ±1. The interaction amplitudes in the DSS theory involve the Regge trajectory α of the exchanged particle which at $t = m_{\text{ex}}^2$ (the square of the mass of the exchanged particle) reduces to the spin of the corresponding physical particle, i.e.,

$$\alpha(m_{\text{ex}}^2) = \text{spin}_{\text{ph}} \tag{1.1}$$

It is shown by Sciarrino and Toller (1966) that at $t = 0$ there exists a relation

$$\alpha_n(0) = \alpha(0) - n - 1 \tag{1.2a}$$

[1] Permanent address: Department of Physics, University of Azarabadgan, Tabriz, Iran.

Each Lorentz pole corresponds to an infinite number of daughter poles.[2] In equation (1.2) $\alpha_n(0)$ is the Regge intercept of the nth daughter at $t = 0$. For the leading Regge trajectory (i.e., for $n = 0$, dropping the subscript n from α_n) we may write

$$\alpha(0) = \sigma(0) - 1 \tag{1.2b}$$

In their paper DSS have anticipated an extension of this result at $t \neq 0$, rewriting the relation as

$$\alpha(t) = \sigma(t) - 1 \tag{1.3}$$

From equations (1.1) and (1.3), for a physical particle, we obtain a relation between σ and its spin as follows:

$$\sigma = \text{spin} + 1 \tag{1.4}$$

Thus in general for a particle, σ has a fixed value. Also from the well-known condition (see, for example, Gelfand et al., 1963).

$$|\dot{J}_0| < |\sigma| \tag{1.5}$$

$\dot{J}_0$ can acquire values consistent with equation (1.5). Hence, the principal series irreducible representations of the homogeneous Lorentz group which enter in describing a particle are determined by the possible pairs of $\dot{J}_0$ and σ.

The basic assumption made by one of us (M.S.) in a previous paper (Komy et al., 1968) is that the exchanged trajectories in the t channel exist in a state (or superposed state) that is a superposition of some basic states called the "Lorentz states." The Lorentz states span a representation space of the Lorentz group, for which its irreducible components operate on the aforementioned Lorentz states and are determined by the different $\dot{J}_0$ values and σ. Thus, if the exchanged trajectories exist in states belonging to the representation space τ, in terms of its irreducible components, τ can be written as follows:

$$\tau = \sum_{\text{allowed } \dot{J}_0 \text{ values}} \oplus \, \tau(\dot{J}_0, \sigma) \tag{1.6}$$

We interpret the above assumption to mean that the system of equations so obtained would consistently describe the fields associated with a particle of a given spin satisfying equation (1.4) together with the relevant subsidiary conditions, if any.

Coming back to the consideration of the particle of spin 1, the pairs $(\dot{J}, \sigma)$, which enter into its description, are given as $(0, 2)$, $(\pm 1, 2)$, which we designate as $\tau_1 \sim (0, 2)$, $\tau_2 \sim (-1, 2)$, and $\dot{\tau}_2 \sim (1, 2)$. Needless to say, the representations τ_2 and $\dot{\tau}_2$ are conjugate to each other. Furthermore, these representations are exactly the same as given by Gelfand and Yaglom (1963) for a particle of spin 1, and have the same interlocking scheme. In this paper we have, therefore, employed the Gelfand–Yaglom technique for obtaining the relevant equations for a particle of spin 1 and mass zero, i.e., a photon, with the essen-

[2] The parent–daughter phenomenon was anticipated in the works of Volkov and Gribov (1963); Domokos and Suranyi (1964); Freedman and Wang (1967); and others.

tial difference that for a complete study of the problem, in conformity with the basic assumption (Komy et al., 1968) we have worked out the relevant matrices in their most general form rather than truncating them as usual by introducing $\bar{\tau}_2$ as an arbitrary degenerate transformation. For the continuity of treatment and to establish our notation in Section 2 we have briefly reviewed the Gelfand-Yaglom method for mass-zero particles. In Section 3 we have obtained Maxwell's equations together with the subsidiary conditions and have discussed their implications. Section 4 exploits the masslessness of a photon to obtain the Lorentz condition and the photon polarization. In Section 5 we have discussed our results.

2. *The Gelfand-Yaglom Method*

A system of first-order equations for a massless particle can be written as

$$L_0 \frac{\partial\Psi}{\partial x_0} + L_1 \frac{\partial\Psi}{\partial x_0} + L_2 \frac{\partial\Psi}{\partial x_2} + L_3 \frac{\partial\Psi}{\partial x_3} = 0 \tag{2.1}$$

where $x_0 \equiv ct$ and the matrices $L_\mu(\mu = 0, 1, 2, 3)$ act in the space of field functions $\psi(x)$, which we designate as the space R. To equation (2.1) we apply a simultaneous transformation,

$$x' = gx, \qquad \psi'(x') = T_g\psi(x) \tag{2.2}$$

where g is any Lorentz transformation of the coordinates and T_g is a representation of the Lorentz group in the space of the ψ functions. Application of equation (2.2) to equation (2.1) reduces it to the following:

$$\sum_{\mu,\nu} L_\mu T_g^{-1} \frac{\partial\Psi'}{\partial x'_\nu} g_{\nu\mu} = 0 \tag{2.3}$$

According to Gelfand and Yaglom, if there exists a nondegenerate transformation V_g such that

$$\sum_{\mu,\nu} V_g L_\mu T_g^{-1} \frac{\partial\Psi'}{\partial x'_\nu} g_{\nu\mu} \equiv \sum_\nu L_\nu \frac{\partial\Psi'(x')}{\partial x'_\nu} \qquad (\text{for all } \psi'(x')) \tag{2.4}$$

i.e.,

$$\sum_\mu V_g L_\mu T_g^{-1} g_{\nu\mu} = L_\nu \tag{2.5}$$

it follows that equations (2.1) and (2.3) are equivalent, which means equation (2.1) is essentially unaltered after the substitution given by equation (2.2). However, for a massive particle for the invariance of equation (2.1) under this transformation it is necessary that $V_g = T_g$. At this juncture, we notice that instead of multiplying equation (2.3) by V_g from the left, we can as well multiply it by T_g from the left. Thus, in conformity with our theory, we multiply equation (2.3) by T_g from the left and obtain the invariance of equation (2.1), which yields

$$\sum_\mu T_g L_\mu T_g^{-1} g_{\mu\nu} = L_\nu \tag{2.6}$$

Now, if the space R in which the representation T_g acts is decomposed into a linear sum of subspaces R^τ in each of which there acts an irreducible representation τ of the proper Lorentz group designated by the pair $\tau \sim (\dot{J}_0 \equiv l_0, \sigma \equiv l_1)$, and if we introduce into each subspace R^τ a canonical basis $\{\xi^\tau_{lm}\}$ ($m = -l_1, -l+1, \ldots, l$; l being an integer or half an odd integer) of the eigenvectors of the usual Lorentz operator H_3 (which is also a generator of the rotation group), following Gelfand et al., it is possible to work out the matrices L_μ under the specific conditions that the pairs (l_0, l_1) and (l'_0, l'_1), defining τ and τ' be interlocking, i.e.,

$$(l'_0, l'_1) = (l_0 \pm 1, l_1)$$

or (2.7)

$$(l'_0, l'_1) = (l_0, l_1 \pm 1)$$

Gelfand and Yaglom have obtained these matrices as follows:

$$L_0 = \| c_l^{\tau\tau'} \delta_{ll'} \delta_{mm'} \| \tag{2.8}$$

where for

$$\begin{aligned} (l'_0, l'_1) &= (l_0 + 1, l_1) \\ c_l^{\tau\tau'} &= c^{\tau\tau}[(l + l_0 + 1)(l - l_0)]^{1/2} \\ c_l^{\tau'\tau} &= c_\tau^{\tau'\tau}[(l + l_0 + 1)(l - l_0)]^{1/2} \end{aligned} \tag{2.9}$$

and for

$$\begin{aligned} (l'_0, l'_1) &= (l_0, l + 1) \\ c_l^{\tau\tau'} &= c^{\tau\tau'}[(l + l_1 + 1)(l - l_1)]^{1/2} \\ c_l^{\tau'\tau} &= c^{\tau'\tau}[(l + l_1 + 1)(l - l_1)]^{1/2} \end{aligned} \tag{2.10}$$

$C^{\tau\tau'}$ and $C^{\tau'\tau}$ are arbitrary complex numbers;

$$L_3 = \| a^{\tau\tau'}_{lm;l'm'} \| \tag{2.11}$$

where

$$\begin{aligned} a^{\tau\tau'}_{lm;l-1,m} &= i[(l^2 - m^2)]^{1/2}[C_l^\tau c_{l-1}^{\tau\tau'} - C_{l+1}^{\tau'} c_l^{\tau\tau'}] \\ a^{\tau\tau'}_{lm;l,m} &= -imC_l^{\tau\tau'}[A_l^\tau - A_l^{\tau'}] \\ a^{\tau\tau'}_{lm;l+1,m} &= -i[(l+1)^2 - m^2]^{1/2}[C_{l+1}^\tau c_{l+1}^{\tau\tau'} - C_{l+1}^{\tau'} c_l^{\tau\tau'}] \end{aligned} \tag{2.12}$$

$$L_1 = \| b^{\tau\tau'}_{lm;l'm'} \|, \qquad L_2 = \| d^{\tau\tau'}_{lm;l'm'} \| \tag{2.13}$$

where

$$
\begin{aligned}
b^{\tau\tau'}_{lm;l-1,m-1} &= -id_{lm;l-1,m-1} \\
&= (i/2)[(l+m)(l+m-1)]^{1/2}[C^{\tau}_{l}c^{\tau\tau'}_{l-1} - C^{\tau'}_{l}c^{\tau\tau'}_{l}] \\
b^{\tau\tau'}_{lm;l-1,m+1} &= id_{lm;l-1,m+1} \\
&= (-i/2)[(l-m)(l-m-1)]^{1/2}[C^{\tau}_{l}c^{\tau\tau'}_{l-1} - C^{\tau'}_{l}c^{\tau\tau'}_{l} \\
b^{\tau\tau'}_{lm;l,m-1} &= -id_{lm;l,m-1} \\
&= (i/2)[(l+m)(l-m+1)]^{1/2}c^{\tau\tau'}_{l}[A^{\tau}_{l} - A^{\tau'}_{l}] \qquad (2.14) \\
b^{\tau\tau'}_{lm;l,m+1} &= id_{lm;l,m+1} \\
&= (i/2)[(l-m)(l+m+l)]^{1/2}c^{\tau\tau'}_{l}[A^{\tau}_{l} - A^{\tau'}_{l}] \\
b^{\tau\tau'}_{lm;l+1,m-1} &= -id_{lm;l+1,m-1} \\
&= (i/2)[(l-m+1)(l-m+2)]^{1/2}[C^{\tau}_{l+1}c^{\tau\tau'}_{l+1} - C^{\tau'}_{l+1}c^{\tau\tau'}_{l}] \\
b^{\tau\tau'}_{lm;l+1,m+1} &= id_{lm;l+1,m+1} \\
&= (-i/2)[(l+m+1)(l+m+2)]^{1/2}[C^{\tau}_{l+1}c^{\tau\tau'}_{l+1} - C^{\tau'}_{l+1}c^{\tau\tau'}_{l}]
\end{aligned}
$$

where

$$
A_l = \frac{il_0 l_1}{l(l+1)}, \qquad C_l = (i/2)[(l^2 - l_0{}^2)(l^2 - l_1{}^2)/(4l^2 - 1)]^{1/2} \qquad (2.15)
$$

Equation (2.14) defines the number C_l to within a sign. In particular, if C_l is real then we regard it to be positive.

3. *Maxwell's Equations*

The representations for the particle with spin 1 and mass zero are given as $\tau_1 \sim (0, 2)$, $\tau_2 \sim (-1, 2)$, $\dot{\tau}_2 \sim (1, 2)$, which have the following interlocking scheme:

$$
\tau_2 \leftrightarrow \tau_1 \leftrightarrow \dot{\tau}_2
$$

The most general form of the matrices L_ν, equation (2.5), having the above interlocking scheme can be worked out in terms of the following basis:

$$
\xi^{\dot{\tau}_2}_{11}, \xi^{\dot{\tau}_2}_{10}, \xi^{\dot{\tau}_2}_{1,-1}, \xi^{\tau_1}_{00}, \xi^{\tau_1}_{11}, \xi^{\tau_1}_{10}, \xi^{\tau_1}_{1,-1}, \xi^{\tau_2}_{11}, \xi^{\tau_2}_{10}, \xi^{\tau_2}_{1,-1}
$$

It can be shown that, if we invoke the invariance of the equations under the complete Lorentz group, the matrix elements of L_μ must satisfy the following restrictions:

$$
c^{\tau_1\dot{\tau}_2} = c^{\tau_1\tau_2} = c_1 \quad \text{and} \quad c^{\dot{\tau}_2\tau_1} = c^{\tau_2\tau_1} = c_2 \qquad (3.1)
$$

Also if we wish to ensure the existence of a Lagrangian leading to the relevant equations, the matrix elements must satisfy the further restrictions

$$
c^{\dot{\tau}_2\tau_1}_{\tau} = \bar{c}^{\tau_1\dot{\tau}_2} \qquad (3.2)
$$

or

$$\bar{c}_1 = c_2 \tag{3.3}$$

For $c_1 = i$, we have arrived at the following matrices for L_0, L_1, L_2, and L_3:

$$L_0 = \left\|\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i \\ \hline 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{array}\right\|, \quad L_1 = \left\|\begin{array}{cccccccccc} 0 & 0 & 0 & -1 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & i & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ i & 0 & i & 0 & 0 & 0 & 0 & -i & 0 & -i \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -i & 0 & 0 & 0 & 0 \end{array}\right\| \tag{3.4}$$

$$L_2 = \left\|\begin{array}{cccccccccc} 0 & 0 & 0 & -i & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline -i & 0 & -i & 0 & 0 & 0 & 0 & -i & 0 & -i \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -i & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right\|, L_3 = \left\|\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ \hline 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{array}\right\|$$

Defining ψ as a ten-component quantity

$$\psi \equiv \left\|\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{array}\right\| \tag{3.5}$$

the substitution of the matrices equation (3.4) and for ψ equation (3.5) into equation (2.1) yields a system of first-order equations, which, with a little algebraic manipulation, can be recast into the following set of equations:

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}F_{01} + \frac{\partial}{\partial x_2}F_{02} + \frac{\partial}{\partial x_3}F_{03} = 0\\
&\frac{\partial}{\partial x_0}F_{30} + \frac{\partial}{\partial x_1}F_{31} + \frac{\partial}{\partial x_2}F_{32} = 0\\
&\frac{\partial}{\partial x_0}F_{20} + \frac{\partial}{\partial x_1}F_{21} + \frac{\partial}{\partial x_3}F_{23} = 0\\
&\frac{\partial}{\partial x_0}F_{10} + \frac{\partial}{\partial x_2}F_{12} + \frac{\partial}{\partial x_3}F_{13} = 0
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
F_{01} &\equiv i(-\phi_1 + \phi_3 - \phi_4 + \phi_6), & F_{10} &\equiv i(\phi_1 - \phi_3 + \phi_4 - \phi_6)\\
F_{02} &\equiv (-\phi_1 - \phi_3 - \phi_4 - \phi_6), & F_{20} &\equiv (\phi_1 + \phi_3 + \phi_4 + \phi_6)\\
F_{03} &\equiv i(\phi_2 + \phi_5), & F_{30} &\equiv i(-\phi_2 - \phi_5)\\
F_{12} &\equiv 2(-\phi_2 + \phi_5), & F_{21} &\equiv 2(\phi_2 - \phi_5)\\
F_{13} &\equiv i(-\phi_1 + \phi_4 - \phi_3 + \phi_6), & F_{31} &\equiv i(\phi_1 + \phi_3 - \phi_4 - \phi_6)\\
F_{23} &\equiv (-\phi_1 + \phi_3 + \phi_4 - \phi_6), & F_{32} &\equiv (+\phi_1 - \phi_3 - \phi_4 + \phi_6)
\end{aligned}
\tag{3.7}
$$

together with

$$
L'_0 \frac{\partial\psi}{\partial x_0} + L'_1 \frac{\partial\psi}{\partial x_1} + L'_2 \frac{\partial\psi}{\partial x_2} + L'_3 \frac{\partial\psi}{\partial x_3} = 0 \tag{3.8}
$$

$$
L''_0 \frac{\partial\psi}{\partial x_0} + L''_1 \frac{\partial\psi}{\partial x_1} + L''_2 \frac{\partial\psi}{\partial x_2} + L''_3 \frac{\partial\psi}{\partial x_3} = 0 \tag{3.9}
$$

where

$$
\begin{aligned}
L'_0 &= \begin{Vmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{Vmatrix}, &
L'_1 &= \begin{Vmatrix} -1 & 0 & i & 0 \\ 0 & i & 0 & i \\ 1 & 0 & i & 0 \end{Vmatrix}\\
L'_2 &= \begin{Vmatrix} -i & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -i & 0 & 1 & 0 \end{Vmatrix}, &
L'_3 &= \begin{Vmatrix} 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{Vmatrix}\\
L''_0 &= \begin{Vmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{Vmatrix}, &
L''_1 &= \begin{Vmatrix} -1 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ 1 & 0 & -i & 0 \end{Vmatrix}\\
L''_2 &= \begin{Vmatrix} -i & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -i & 0 & -1 & 0 \end{Vmatrix}, &
L''_3 &= \begin{Vmatrix} 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{Vmatrix}
\end{aligned}
\tag{3.10}
$$

According to our interpretation of the basic assumption, the equations would consistently describe the field equations along with the relevant subsidiary conditions, if any, describing a photon. Equation (3.6) together with equation (3.7) we immediately recognize as Maxwell's equations in their covariant form. Just these equations would have been obtained if we had used equation (2.1) with the bracketed portions of the matrices L_0, L_1, L_2, L_3, and ψ, which we may designate as $[L_0], [L_1], [L_2], [L_3]$, and $[\psi]$. Thus equations (3.8) and (3.9) would constitute the subsidiary conditions.

To extract information from these equations we proceed as follows. We expand the solution of equations (3.8) and (3.9) in a four-dimensional Fourier integral

$$\psi(x) = \int u(P) e^{iP_\mu x_\mu}\, dP. \qquad (\mu = 0, 1, 2, 3) \tag{3.11}$$

where

$$P_\mu x_\mu = -P_0 x_0 + \bar{P}\cdot\bar{x} \quad \text{and} \quad dP = dP_0\, dP_1\, dP_2\, dP_3 = d\mathbf{P}\, dP_0 \tag{3.12}$$

Equation (3.11) can be considered as a wave packet that is a superposition of waves with the wave vectors P_μ which, following the deBroglie relation, can be associated with free particles of momentum (in units of $c = \hbar = 1$) **P** and energy $P_0 = E$.

Inserting equation (3.11) into equations (3.8) and (3.9), we obtain the following set of equations:

$$-(P_1 + iP_2)u_1(P) + i(P_0 - P_3)u_2(P) + (iP_1 - P_2)u_3(P) = 0 \tag{3.13a}$$

$$-P_3 u_1(P) + (iP_1 + P_2)u_2(P) + iP_0 u_3(P) + (iP_1 - P_2)u_4(P) = 0 \tag{3.13b}$$

$$(P_1 - iP_2)u_1(P) + (iP_1 + P_2)u_3(P) + i(P_0 + P_3)u_4(P) = 0 \tag{3.13c}$$

$$-(P_1 + iP_2)u_1(P) + i(P_0 + P_3)u_2(P) + (-iP_1 + P_2)u_3(P) = 0 \tag{3.14a}$$

$$-P_3 u_1(P) - (iP_1 + P_2)u_2(P) + iP_0 u_3(P) - (iP_1 - P_2)u_4(P) = 0 \tag{3.14b}$$

$$(P_1 - iP_2)u_1(P) - (iP_1 + P_2)u_3(P) + i(P_0 - P_3)u_4(P) = 0 \tag{3.14c}$$

Now adding equations (3.13b) and (3.14b) and subtracting equations (3.13c) and (3.14c) we can recast the above sets in a matrix equation as below:

$$\begin{Vmatrix} -(P_1 + iP_2) & i(P_0 - P_3) & (iP_1 - P_2) & 0 \\ -2P_3 & 0 & 2iP_0 & 0 \\ -(P_1 + iP_2) & i(P_0 + P_3) & (-iP_1 + P_2) & 0 \\ 0 & 0 & 2(iP_1 + P_2) & 2iP_3 \end{Vmatrix} \begin{vmatrix} u_1(P) \\ u_2(P) \\ u_3(P) \\ u_4(P) \end{vmatrix} = 0 \tag{3.15}$$

Defining the 4 x 4 matrix in equation (3.15) as $\mathscr{L}(P)$ and the column vector as $u(P)$, we may write equation (3.15) as

$$\mathscr{L}(P)u(P) = 0 \tag{3.16}$$

Equation (3.16) admits of a non zero solution only for the condition that

$$\det |\mathscr{L}(P)| = 0 \tag{3.17}$$

For $P_1 \neq 0, P_2 \neq 0$, and $P_3 \neq 0$ equation (3.17) can be recast as the difference of two equal terms, i.e.,

$$P_3(P_1^2 + P_2^2)(P_0^2 - P_3^2 - P_2^2 - P_1^2) - P_3(P_1^2 + P_2^2)(P_0^2 - P_3^2 - P_2^2 - P_1^2) = 0 \tag{3.18}$$

Equation (3.18) admits of interesting physical interpretations for different possibilities. Barring the trivial case $P_1 = P_2 = P_3 = 0$ (which we have excluded ourselves), the first possibility is that each of the expressions in equation (3.18) is zero, which means

$$P_0^2 - P_3^2 - P_2^2 - P_1^2 = 0 \tag{3.19}$$

The obvious interpretation of equation (3.19) is that the rest mass of the particles associated with any of the plane waves with the four-momentum $P(P_0, P_1, P_2, P_3)$ is zero; the inequalities amount to saying that such a particle is never at rest. These are exactly the properties which we associate with a photon. Again barring the trivial case mentioned above, the second possibility is that the expression $P_3(P_1^2 + P_2^2)(P_0^2 - P_1^2 - P_2^2 - P_3^2)$ is not equal to zero, and only the difference of two such expressions [i.e., equation (3.18)] equals zero. In view of the fact that one of the expressions occurs with a negative sign and is obtainable by a replacement of the four-momentum $P_\mu \to -P_\mu$ or explicitly $P_1 \to -P_1, P_2 \to -P_2, P_3 \to -P_3$, and $P_0 \to -P_0$, we may regard equation (3.18) as indicative of a massive particle and its antiparticle to be existing at the same space-time point, which means the annihilation of a massive particle and its antiparticle emerges from equation (3.18) as a possible condition for the creation of a photon. Furthermore, from the point of view of the surfaces of transitivity for the Lorentz group in the momentum space, the first expression in equation (3.18),

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 = \text{const} > 0, \qquad P_0 > 0 \tag{3.20}$$

corresponds to the upper branch of a hyperboloid of two sheets, and the other,

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 = \text{const} > 0, \qquad P_0 < 0 \tag{3.21}$$

to its lower branch. Equations (3.20) and (3.21) tell us that the annihilating system consisted of particles whose four-momenta lie on the upper and lower branch of the mass-shell hyperboloid.

4. *Exploitation of Photon Masslessness*

Rewriting equation (3.6) as

$$\frac{\partial F_{\mu\nu}}{\partial x^\mu} = 0 \qquad (\mu = 0, 1, 2, 3) \tag{4.1}$$

and defining as usual

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \qquad \Box = \frac{\partial^2}{\partial x^{02}} - \nabla^2 \tag{4.2}$$

equation (4.1) can be recast as

$$\Box A_\mu - \frac{\partial}{\partial x^\nu}\left(\frac{\partial A_\mu}{\partial x^\mu}\right) = 0 \tag{4.3}$$

Equation (4.3) along with the subsidiary condition, i.e., equation (3.19), which led us to the properties associated with a photon, automatically leads us to the Lorentz condition on the four potentials A_μ:

$$\frac{\partial A_\mu}{\partial x^\mu} = 0 \tag{4.4}$$

In addition, if we select the x_3 axis along the positive direction of the momentum of a photon, in such a coordinate system the photon energy momentum four-vector P has the components $P(P_0, 0, 0, P_3)$. The plane-wave solutions of Maxwell's equations in their covariant form satisfy the equation

$$(P_3[L_3] - P_0[L_0])F_{\mu\nu}(P) = 0 \tag{4.5}$$

Which means that $F_{\mu\nu}(P)$ belongs to the subspace, annihilating the matrix $P_3[L_3] - P_0[L_0]$.

The matrix $P_3[L_3] - P_0[L_0]$ written in full reads

$$P_3[L_3] - P_0[L_0] = \begin{array}{c} \begin{array}{cccccc} \xi^{\dot{\tau}_2}_{11} & \xi^{\dot{\tau}_2}_{10} & \xi^{\dot{\tau}_2}_{1-1} & \xi^{\tau_2}_{11} & \xi^{\tau_2}_{10} & \xi^{\tau_2}_{1-1} \end{array} \\ \left\| \begin{array}{cccccc} 0 & P_3 & 0 & 0 & P_3 & 0 \\ -i(P_3 + P_0) & 0 & 0 & i(P_3 - P_0) & 0 & 0 \\ 0 & -iP_0 & 0 & 0 & -iP_0 & 0 \\ 0 & 0 & i(P_3 - P_0) & 0 & 0 & -i(P_3 + P_0) \end{array} \right\| \end{array} \tag{4.6}$$

The first row of the matrix is proportional to the third. Thus, we can add further rows to this matrix by adding and subtracting the first and the third rows and write these explicitly to give the following 6 x 6 square matrix:

$$\begin{array}{c} \begin{array}{cccccc} \xi^{\dot{\tau}_2}_{11} & \xi^{\dot{\tau}_2}_{10} & \xi^{\dot{\tau}_2}_{1-1} & \xi^{\tau_2}_{11} & \xi^{\tau_2}_{10} & \xi^{\tau_2}_{1-1} \end{array} \\ \left\| \begin{array}{cccccc} 0 & P_3 & 0 & 0 & P_3 & 0 \\ -i(P_3 + P_0) & 0 & 0 & i(P_3 - P_0) & 0 & 0 \\ 0 & -iP_0 & 0 & 0 & -iP_0 & 0 \\ 0 & 0 & i(P_3 - P_0) & 0 & 0 & -i(P_3 + P_0) \\ 0 & (P_3 + iP_0) & 0 & 0 & (P_3 + iP_0) & 0 \\ 0 & (P_3 - iP_0) & 0 & 0 & (P_3 - iP_0) & 0 \end{array} \right\| \end{array} \tag{4.7}$$

Equation (3.19), in this co-ordinate system reduces to

$$P_0{}^2 - P_3{}^2 = 0 \tag{4.8}$$

yielding

$$P_0 = \pm P_3 \tag{4.9}$$

Now, for $P_0 = P_3 \neq 0$ the subspace $R(P_3, P_3)$ is two-dimensional and contains the vectors $\xi^{\tau_2}_{1\dot{1}}$ and $\xi^{\dot{\tau}_2}_{11}$. In the case $P_0 = -P_3 \neq 0$, again the annihilating subspace is two dimensional and contains the vectors $\xi^{\dot{\tau}_2}_{1-1}$ and $\xi^{\tau_2}_{1\dot{1}}$. In both these cases the annihilating subspaces are invariant under the operator H_3 with the eigenvalues of this operator 1 and -1. The polarization value $m = 0$ is excluded since the linear combination of the vectors $\xi^{\dot{\tau}_2}_{1\dot{0}}$, $\xi^{\tau_2}_{10}$ can belong to the annihilating subspaces of the matrix only if $P_3 = P_0 = 0$. Thus the polarization of the photon can assume olny two values: 1, -1.

5. *Discussion of Results*

In this paper, to the assumption made by one of us (M.S.) offering an interpretation, we have applied it to a particle of spin 1 and mass zero. Our interpretation has allowed us to introduce $\dot{\tau}_2 \sim (0, 2)$ as an integral part of the theory, contrary to its usual arbitrary introduction as a degenerate transformation. The most general form of the matrices to which we were led by the theory yielded a subsidiary condition which admitted of two possibilities in interpreting it: (i) The photon is never at rest, and if, however, it were to be brought to rest, its rest mass would be zero; (ii) the annihilation of a massive particle and its antiparticle can create a photon. Furthermore, we have shown that the condition of masslessness of a photon with the covariant Maxwell's equations results in the Lorentz condition, also, that the present formulation yields correct values of photon polarization.

Acknowledgments

One of us (M.S.) wishes to thank Professor H. D. Doebner and Dr. M. Bakri for interesting discussions on this work. He is grateful to Professors Abdus Salam and P. Budini, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

Delbourgo, R., Salam, A., and Strathdee, J. (1967). *Physics Letters*, **25B**, 230.
Domokos, G., and Suranyi, P. (1964). *Nuclear Physics*, **54**, 529.
Freedman, D. Z., and Wang, J. M. (1967). *Physical Review*, **153B**, 1596.
Gelfand, I. M., Minlos, R. A., and Shapiro, Ya. (1963). *Representations of the Rotation and Lorentz Groups and Their Applications*, Pergamon Press, New York.
Komy, S. R., Samiullah, M., and Mahanta, P. (1968). *Nuovo Cimento*, **55A**, 423.
Raghavan, S., and Samiullah, M. (1969). *Nuclear Physics*, **B10**, 48.
Sciarrino, A., and Toller, M. (1966). Istituto di Fisica "G. Marconi", Università di Roma, Nota Interna No. 108.
Volkov, R. J. N., and Gribov, V. N. (1963). *Soviet Physics–JETP*, **17**, 720.